

UNDERSTANDING PROGRAM EFFICIENCY: 2

(download slides and .py files and follow along!)

6.0001 LECTURE 11

TODAY

- Classes of complexity
- Examples characteristic of each class

WHY WE WANT TO UNDERSTAND EFFICIENCY OF PROGRAMS

- how can we reason about an algorithm in order to predict the amount of time it will need to solve a problem of a particular size?
- how can we relate choices in algorithm design to the time efficiency of the resulting algorithm?
 - are there fundamental limits on the amount of time we will need to solve a particular problem?

ORDERS OF GROWTH: RECAP

Goals:

- want to evaluate program's efficiency when **input is very big**
- want to express the **growth of program's run time** as input size grows
- want to put an **upper bound** on growth – as tight as possible
- do not need to be precise: **“order of” not “exact”** growth
- we will look at **largest factors** in run time (which section of the program will take the longest to run?)
- **thus, generally we want tight upper bound on growth, as function of size of input, in worst case**

COMPLEXITY CLASSES: RECAP

- 
- $O(1)$ denotes constant running time
 - $O(\log n)$ denotes logarithmic running time
 - $O(n)$ denotes linear running time
 - $O(n \log n)$ denotes log-linear running time
 - $O(n^c)$ denotes polynomial running time (c is a constant)
 - $O(c^n)$ denotes exponential running time (c is a constant being raised to a power based on size of input)

COMPLEXITY CLASSES ORDERED LOW TO HIGH

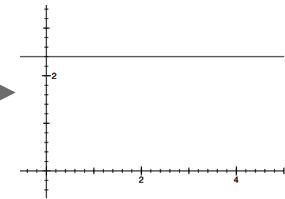


*c is a
constant*

$O(1)$

:

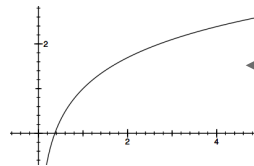
constant



$O(\log n)$

:

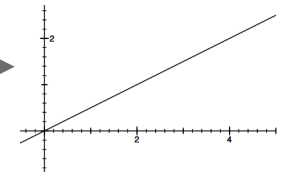
logarithmic



$O(n)$

:

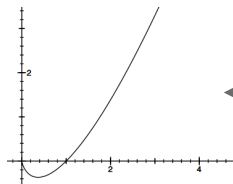
linear



$O(n \log n)$

:

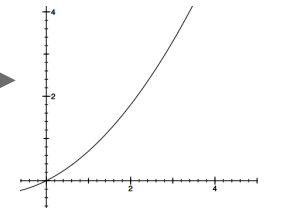
loglinear



$O(n^c)$

:

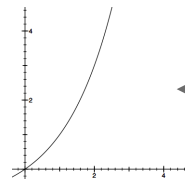
polynomial



$O(c^n)$

:

exponential



COMPLEXITY GROWTH

CLASS	n=10	= 100	= 1000	= 1000000
O(1)	1	1	1	1
O(log n)	1	2	3	6
O(n)	10	100	1000	1000000
O(n log n)	10	200	3000	6000000
O(n^2)	100	10000	1000000	1000000000000
O(2^n)	1024	12676506 00228229 40149670 3205376	1071508607186267320948425049060 0018105614048117055336074437503 8837035105112493612249319837881 5695858127594672917553146825187 1452856923140435984577574698574 8039345677748242309854210746050 6237114187795418215304647498358 1941267398767559165543946077062 9145711964776865421676604298316 52624386837205668069376	Good luck!!



CONSTANT COMPLEXITY

- complexity independent of inputs
- very few interesting algorithms in this class, but can often have pieces that fit this class
- can have loops or recursive calls, but ONLY IF number of iterations or calls independent of size of input

LOGARITHMIC COMPLEXITY

- complexity grows as log of size of one of its inputs
- example:
 - bisection search
 - binary search of a list

BISECTION SEARCH

- suppose we want to know if a particular element is present in a list
- saw last time that we could just “walk down” the list, checking each element
- complexity was linear in length of the list
- suppose we know that the list is ordered from smallest to largest
 - saw that sequential search was still linear in complexity
 - can we do better?

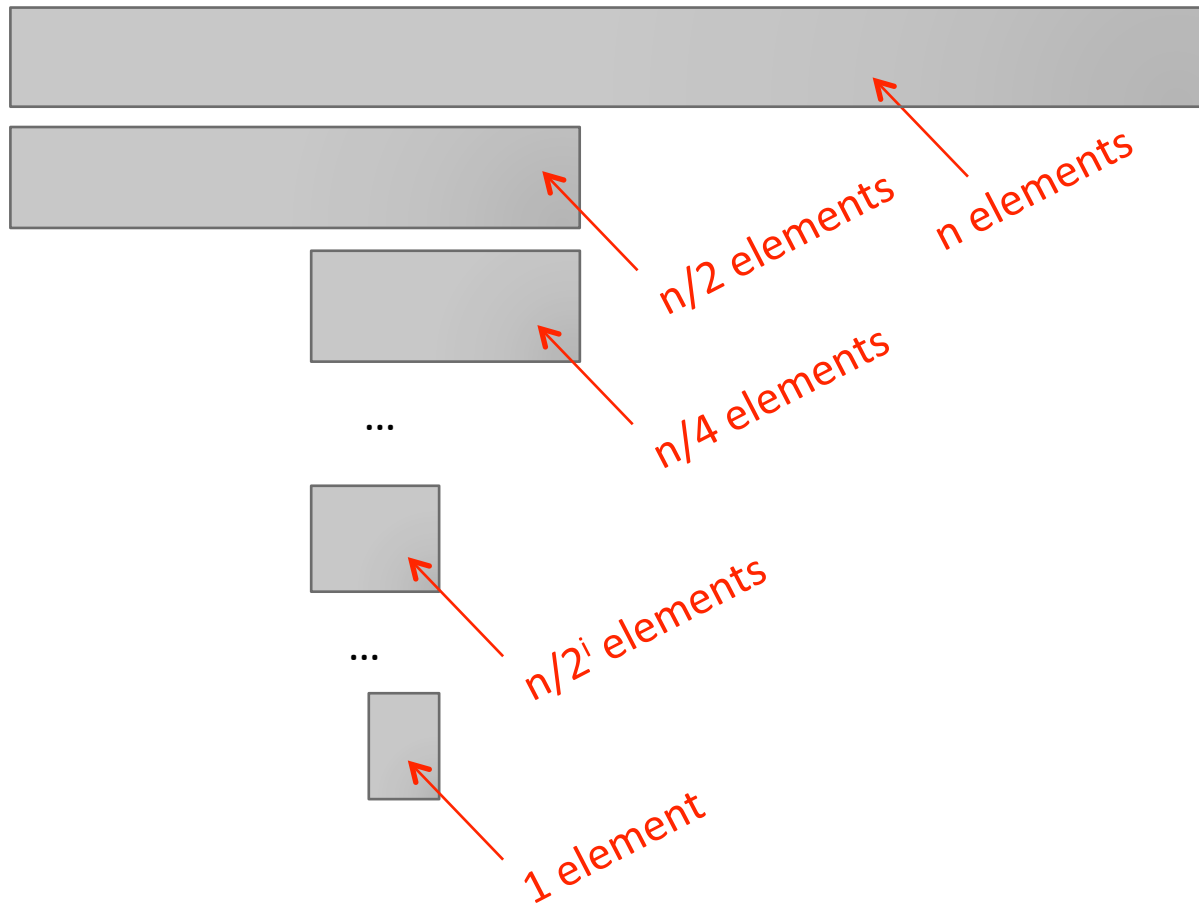
BISECTION SEARCH

1. pick an index, i , that divides list in half
2. ask if $L[i] == e$
3. if not, ask if $L[i]$ is larger or smaller than e
4. depending on answer, search left or right half of L for e

A new version of a divide-and-conquer algorithm

- break into smaller version of problem (smaller list), plus some simple operations
- answer to smaller version is answer to original problem

BISECTION SEARCH COMPLEXITY ANALYSIS



- finish looking through list when

$$1 = n/2^i$$

$$\text{so } i = \log n$$

- complexity of recursion is **$O(\log n)$** – where n is $\text{len}(L)$

BISECTION SEARCH IMPLEMENTATION 1

```
def bisect_search1(L, e):
```

```
    if L == []:
```

```
        return False
```

*constant
O(1)*

```
    elif len(L) == 1:
```

```
        return L[0] == e
```

*constant
O(1)*

```
    else:
```

```
        half = len(L)//2
```

*constant
O(1)*

```
        if L[half] > e:
```

```
            return bisect_search1(L[:half], e)
```

*NOT constant,
copies list*

```
        else:
```

```
            return bisect_search1(L[half:], e)
```

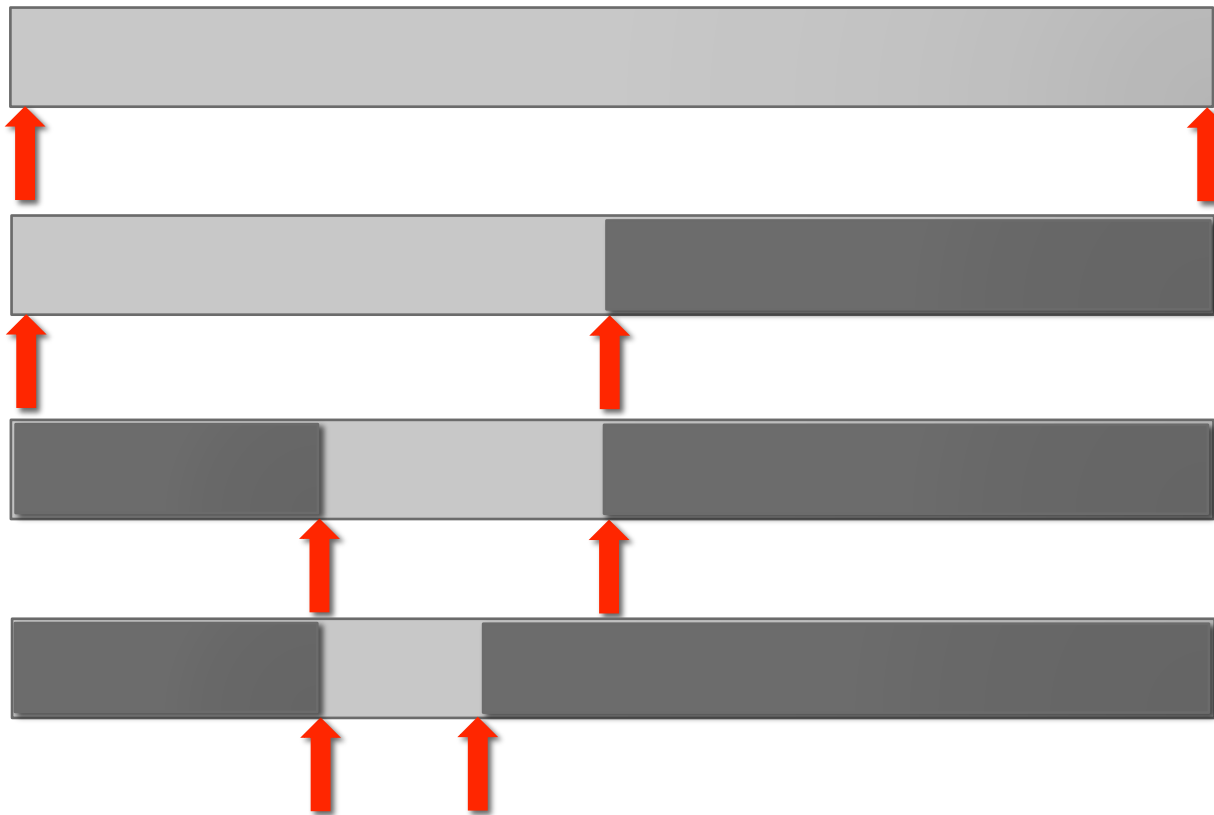
*NOT constant
NOT constant*

COMPLEXITY OF FIRST BISECTION SEARCH METHOD

■ **implementation 1 – bisect_search1**

- $O(\log n)$ bisection search calls
 - On each recursive call, size of range to be searched is cut in half
 - If original range is of size n , in worst case down to range of size 1 when $n/(2^k) = 1$; or when $k = \log n$
- $O(n)$ for each bisection search call to copy list
 - This is the cost to set up each call, so do this for each level of recursion
- $O(\log n) * O(n) \rightarrow \mathbf{O(n \log n)}$
- if we are really careful, note that length of list to be copied is also halved on each recursive call
 - turns out that total cost to copy is $\mathbf{O(n)}$ and this dominates the $\log n$ cost due to the recursive calls

BISECTION SEARCH ALTERNATIVE



- still reduce size of problem by factor of two on each step
- but just keep track of low and high portion of list to be searched
- avoid copying the list
- complexity of recursion is again $O(\log n)$ – where n is $\text{len}(L)$

BISECTION SEARCH IMPLEMENTATION 2

```
def bisect_search2(L, e):  
    def bisect_search_helper(L, e, low, high):  
        if high == low:  
            return L[low] == e  
        mid = (low + high)//2  
        if L[mid] == e:  
            return True  
        elif L[mid] > e:  
            if low == mid: #nothing left to search  
                return False  
            else:  
                return bisect_search_helper(L, e, low, mid - 1)  
        else:  
            return bisect_search_helper(L, e, mid + 1, high)  
    if len(L) == 0:  
        return False  
    else:  
        return bisect_search_helper(L, e, 0, len(L) - 1)
```

*constant other
than recursive call*

*constant other
than recursive call*

COMPLEXITY OF SECOND BISECTION SEARCH METHOD

- **implementation 2 – bisect_search2** and its helper
 - $O(\log n)$ bisection search calls
 - On each recursive call, size of range to be searched is cut in half
 - If original range is of size n , in worst case down to range of size 1 when $n/(2^k) = 1$; or when $k = \log n$
 - pass list and indices as parameters
 - list never copied, just re-passed as a pointer
 - thus $O(1)$ work on each recursive call
 - $O(\log n) * O(1) \rightarrow \mathbf{O(\log n)}$

LOGARITHMIC COMPLEXITY

```
def intToStr(i):  
    digits = '0123456789'  
    if i == 0:  
        return '0'  
    result = ''  
    while i > 0:  
        result = digits[i%10] + result  
        i = i//10  
    return result
```

LOGARITHMIC COMPLEXITY

```
def intToStr(i):  
    digits = '0123456789'  
    if i == 0:  
        return '0'  
    res = ''  
    while i > 0:  
        res = digits[i%10] + res  
        i = i//10  
    return result
```

only have to look at loop as
no function calls

within while loop, constant
number of steps

how many times through
loop?

- how many times can one
divide i by 10?
- $O(\log(i))$

LINEAR COMPLEXITY

- saw this last time
 - searching a list in sequence to see if an element is present
 - iterative loops

O() FOR ITERATIVE FACTORIAL

- complexity can depend on number of iterative calls

```
def fact_iter(n):  
    prod = 1  
    for i in range(1, n+1):  
        prod *= i  
    return prod
```

- overall **$O(n)$** – n times round loop, constant cost each time

O() FOR RECURSIVE FACTORIAL

```
def fact_recur(n):  
    """ assume n >= 0 """  
    if n <= 1:  
        return 1  
    else:  
        return n*fact_recur(n - 1)
```

- computes factorial recursively
- if you time it, may notice that it runs a bit slower than iterative version due to function calls
- still **$O(n)$** because the number of function calls is linear in n , and constant effort to set up call
- **iterative and recursive factorial** implementations are the **same order of growth**

LOG-LINEAR COMPLEITY

- many practical algorithms are log-linear
- very commonly used log-linear algorithm is merge sort
- will return to this next lecture

POLYNOMIAL COMPLEXITY

- most common polynomial algorithms are quadratic, i.e., complexity grows with square of size of input
- commonly occurs when we have nested loops or recursive function calls
- saw this last time

EXPONENTIAL COMPLEXITY

- recursive functions where more than one recursive call for each size of problem
 - Towers of Hanoi
- many important problems are inherently exponential
 - unfortunate, as cost can be high
 - will lead us to consider approximate solutions as may provide reasonable answer more quickly

COMPLEXITY OF TOWERS OF HANOI

- Let t_n denote time to solve tower of size n

- $t_n = 2t_{n-1} + 1$

- $= 2(2t_{n-2} + 1) + 1$

- $= 4t_{n-2} + 2 + 1$

- $= 4(2t_{n-3} + 1) + 2 + 1$

- $= 8t_{n-3} + 4 + 2 + 1$

- $= 2^k t_{n-k} + 2^{k-1} + \dots + 4 + 2 + 1$

- $= 2^{n-1} + 2^{n-2} + \dots + 4 + 2 + 1$

- $= 2^n - 1$

- so order of growth is $O(2^n)$

Geometric growth

$$a = 2^{n-1} + \dots + 2 + 1$$

$$2a = 2^n + 2^{n-1} + \dots + 2$$

$$a = 2^n - 1$$

EXPONENTIAL COMPLEXITY

- given a set of integers (with no repeats), want to generate the collection of all possible subsets – called the power set
- $\{1, 2, 3, 4\}$ would generate
 - $\{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$
- order doesn't matter
 - $\{\}, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$

POWER SET – CONCEPT

- we want to generate the power set of integers from 1 to n
- assume we can generate power set of integers from 1 to $n-1$
- then all of those subsets belong to bigger power set (choosing not include n); and all of those subsets with n added to each of them also belong to the bigger power set (choosing to include n)
- $\{\}, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$
- nice recursive description!

EXPONENTIAL COMPLEXITY

```
def genSubsets(L):  
    res = []  
    if len(L) == 0:  
        return [[]] #list of empty list  
    smaller = genSubsets(L[:-1]) # all subsets without  
last element  
    extra = L[-1:] # create a list of just last element  
    new = []  
    for small in smaller:  
        new.append(small+extra) # for all smaller  
solutions, add one with last element  
    return smaller+new # combine those with last  
element and those without
```

EXPONENTIAL COMPLEXITY

```
def genSubsets(L):  
    res = []  
    if len(L) == 0:  
        return [[]]  
    smaller = genSubsets(L[:-1])  
    extra = L[-1:]  
    new = []  
    for small in smaller:  
        new.append(small+extra)  
    return smaller+new
```

assuming append is
constant time

time includes time to solve
smaller problem, plus time
needed to make a copy of
all elements in smaller
problem

EXPONENTIAL COMPLEXITY

```
def genSubsets(L):  
    res = []  
    if len(L) == 0:  
        return [[]]  
    smaller = genSubsets(L[:-1])  
    extra = L[-1:]  
    new = []  
    for small in smaller:  
        new.append(small+extra)  
    return smaller+new
```

but important to think
about size of smaller

know that for a set of size
 k there are 2^k cases

how can we deduce
overall complexity?

EXPONENTIAL COMPLEXITY

- let t_n denote time to solve problem of size n
- let s_n denote size of solution for problem of size n
- $t_n = t_{n-1} + s_{n-1} + c$ (where c is some constant number of operations)
- $t_n = t_{n-1} + 2^{n-1} + c$
- $= t_{n-2} + 2^{n-2} + c + 2^{n-1} + c$
- $= t_{n-k} + 2^{n-k} + \dots + 2^{n-1} + kc$
- $= t_0 + 2^0 + \dots + 2^{n-1} + nc$
- $= 1 + 2^n + nc$

Thus
computing
power set is
 $O(2^n)$

COMPLEXITY CLASSES

- $O(1)$ – code does not depend on size of problem
- $O(\log n)$ – reduce problem in half each time through process
- $O(n)$ – simple iterative or recursive programs
- $O(n \log n)$ – will see next time
- $O(n^c)$ – nested loops or recursive calls
- $O(c^n)$ – multiple recursive calls at each level

SOME MORE EXAMPLES OF ANALYZING COMPLEXITY

COMPLEXITY OF ITERATIVE FIBONACCI

```
def fib_iter(n):
```

```
    if n == 0:  
        return 0  
    elif n == 1:  
        return 1
```

```
    else:
```

```
        fib_i = 0  
        fib_ii = 1
```

```
        for i in range(n-1):  
            tmp = fib_i  
            fib_i = fib_ii  
            fib_ii = tmp + fib_i
```

```
        return fib_ii
```

constant
 $O(1)$

constant
 $O(1)$

linear
 $O(n)$

constant
 $O(1)$

- Best case:

$O(1)$

- Worst case:

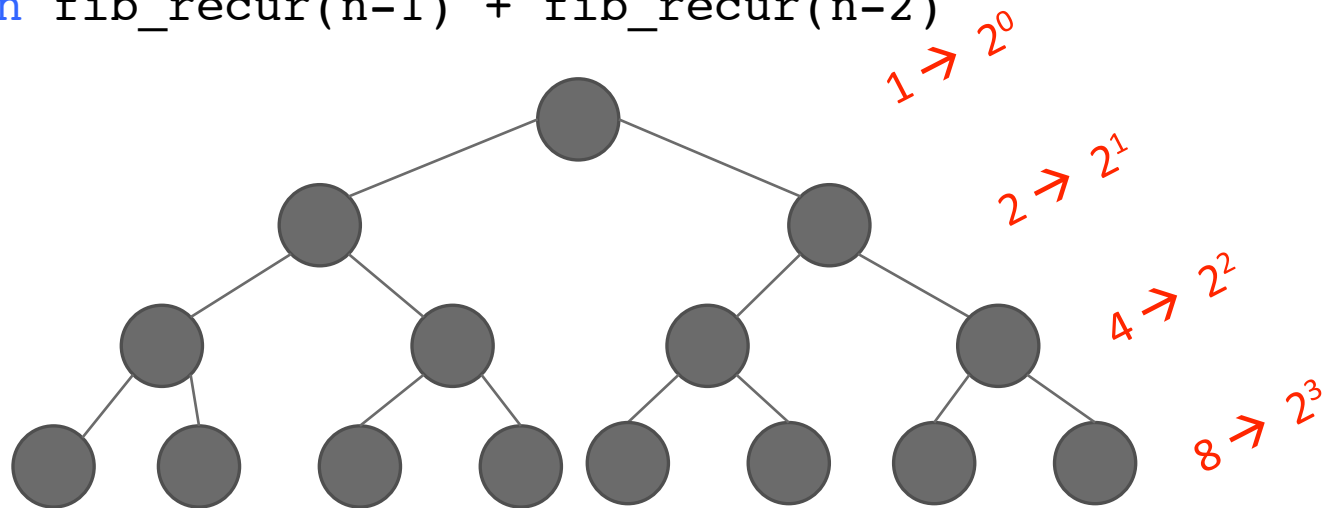
$O(1) + O(n) + O(1) \rightarrow \mathbf{O(n)}$

COMPLEXITY OF RECURSIVE FIBONACCI

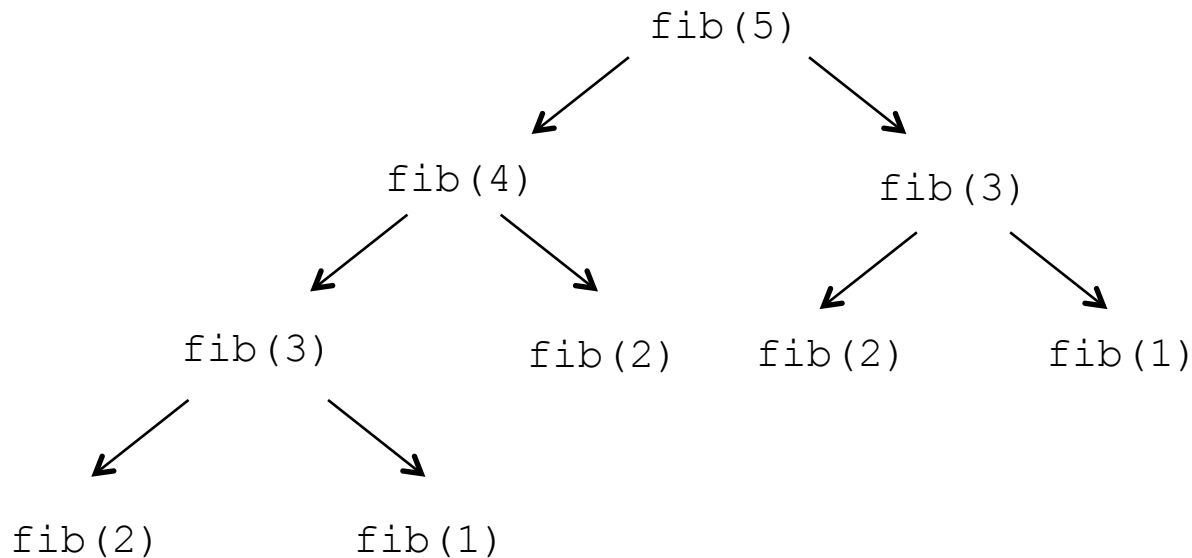
```
def fib_recur(n):  
    """ assumes n an int >= 0 """  
    if n == 0:  
        return 0  
    elif n == 1:  
        return 1  
    else:  
        return fib_recur(n-1) + fib_recur(n-2)
```

- Worst case:

$O(2^n)$



COMPLEXITY OF RECURSIVE FIBONACCI



- actually can do a bit better than 2^n since tree of cases thins out to right
- but complexity is still exponential

BIG OH SUMMARY

- compare **efficiency of algorithms**
 - notation that describes growth
 - **lower order of growth** is better
 - independent of machine or specific implementation
- use Big Oh
 - describe order of growth
 - **asymptotic notation**
 - **upper bound**
 - **worst case** analysis

COMPLEXITY OF COMMON PYTHON FUNCTIONS

■ Lists: n is `len(L)`

- index $O(1)$
- store $O(1)$
- length $O(1)$
- append $O(1)$
- `==` $O(n)$
- remove $O(n)$
- copy $O(n)$
- reverse $O(n)$
- iteration $O(n)$
- in list $O(n)$

■ Dictionaries: n is `len(d)`

■ worst case

- index $O(n)$
- store $O(n)$
- length $O(n)$
- delete $O(n)$
- iteration $O(n)$

■ average case

- index $O(1)$
- store $O(1)$
- delete $O(1)$
- iteration $O(n)$

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